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Structure of the density matrix providing the minimum generalized uncertainty relation for mixed states

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Abstract. For a configurational space of arbitrary dimension a strict form of the uncertainty principle has been obtained, which takes into account the dependence of the inequality limit on the effective number of pure states present in a given statistical mixture. It is shown that in a state with minimal uncertainty the density operator eigenfunctions coincide with the stationary wavefunctions of a multi-dimensional harmonic oscillator. The spectrum of eigenvalues of the density matrix is degenerate and its terms with the same uncertainty have identical weights in the expansion. This leads to additional ‘packing of states’ (i.e. decreasing of the specific phase volume per particular state in the mixture as compared with the limit of the system in a pure state).

The analogy between the generalized uncertainty principle and one of the main postulates of statistical physics on phase-space partitioning into cells corresponding to one quantum state is discussed.

1. Introduction

Uncertainty relations play an ultimate role in the foundation of quantum mechanics [1, 2]. The uncertainty principle states that in an ensemble (which can be described by a vector of the quantum state or by a density matrix) of quantum systems, non-commuting observables cannot be measured with arbitrary accuracy, but there is a lower limit on the product of the variances of such observables. These observables may be the position and momentum of particles, different components of spin or angular momentum, and so forth (see [3] for a comprehensive review). Then, if one describes a system which is in the mixed state, the situation is changed radically—mixed states in general cannot achieve the minimum of uncertainty principle [4]. As shown in papers [5–7], the generalization of the conventional position–momentum uncertainty relation to the case of the density matrix $\rho(x, x')$ gives that the smallest allowable room Δv_p , occupied by the system in phase space, grows proportionally to the effective number N_{eff} of pure states by whose statistical mixture the given density matrix is represented, that is

$$\min(\Delta v_p) \propto N_{\text{eff}}. \quad (1)$$

One of the possible variants of relation (1), which can be used in a real or configurational space $\mathbf{X} = (x_1, \dots, x_s)$ with an arbitrary number of dimensions s , has been obtained in [10] and is of the form

$$\frac{(\Delta x \Delta q)^s}{N_{\text{eff}}} \geq C(s) \left(\frac{1}{2}\right)^s. \quad (2)$$

† Late.

The following definitions are introduced here: Δx , Δq the root mean square widths (additionally averaged over all s degrees of freedom)[†] correspond to the coordinate and wavevector distributions in the state $\rho(x, x')$

$$(\Delta x)^2 = \frac{1}{s} \int d^s X X^2 \rho(\mathbf{X}, \mathbf{X}) \quad (3)$$

$$(\Delta q)^2 = -\frac{1}{s} \int d^s X \int d^s X' \delta(\mathbf{X} - \mathbf{X}') \nabla_{\mathbf{X}}^2 \rho(\mathbf{X}, \mathbf{X}') \quad \left[\nabla_{\mathbf{X}}^2 = \sum_{i=1}^s \partial_{x_i}^2 \right] \quad (4)$$

$$N_{\text{eff}} = \left(\int d^s X \int d^s X' |\rho(\mathbf{X}, \mathbf{X}')|^2 \right)^{-1}. \quad (5)$$

In addition, it is assumed that the normalization condition

$$\int d^s X \rho(\mathbf{X}, \mathbf{X}) = 1 \quad (6)$$

is met and the mean values of position and momentum are equal to zero (without loss of generality the latter can be achieved by choosing an appropriate reference frame). For convenience the right-hand side of inequality (2) is represented as two factors. one of which, $(1/2)^s$, is the minimum of the usual uncertainty relation for pure states and, as is known, is attained for Gaussian wavepackets [1]. The second factor, $C(s)$, can be named the ‘packing coefficient’ and expresses the specificity of the N_{eff} definition and, properly, the influence of the space multi-dimensionality [10]:

$$C(s) = \frac{2^{s+1}(s+1)!}{(s+2)^{s+1}}. \quad (7)$$

The particular interest of relation (2) is caused by its analogy with one of the basic statements of statistical mechanics [11, 12] concerning the partitioning of the phase space into cells, with each of them corresponding to one quantum state. Indeed, the quantity $\Delta v_p = (\Delta x \Delta q)^s$ is a peculiar kind of measure of a system phase volume, whereas N_{eff} is a characteristic number of its possible states. It is therefore obvious that, despite the considerable differences in the formulations used, in either case one can speak about two possible ways of exhibiting one and the same fundamental property of quantum-mechanical objects.

On the other hand, inequality (2) holds not only for the density matrix, but also for correlation functions of the wavefields of various nature, and it can be seminally applied, after appropriate terminology corrections, in other fields of physics. In fact, the basic works [5–7], where the one-dimensional case of relation (2) was obtained, deal with partially coherent light beams.

In [10] a variant of the proof was based on the Carlson-type integral inequalities for the Wigner function. This implies some disadvantages, in particular because such an approach does not determine the explicit form of the density matrix minimizing (2). This question is of interest in itself and, at the same time, is important for applications (for example coherence theory [13]). Hence in this paper we shall consider an alternative way of deriving relation (2). The aim is not only to solve the problem stated, but also to reinforce the inequality (that is to find a more tight and accurate bound) by taking into account the dependence of the packing coefficient on the effective number of states $C(s) \Rightarrow C(s, N_{\text{eff}})$.

[†] Usually (see, e.g., [8, 9]), when considering the uncertainty principle for multi-dimensional cases, one uses a determinant of the covariance matrix. Our approach is based on a trace of such a matrix. But, at a symmetrical minimum uncertainty state—that is when uncertainties in x_i for all i are equal (and similarly for the momentum) and close to the uncertainty minimum—both approaches give the same result.

2. Formulation of the problem and rigorous solution

The method to be applied constitutes a modification of the approach which was developed in [3] to analyse an analogous one-dimensional problem. It is based on the use of standard Lagrange procedure of search for the minimum of the uncertainty volume of the system (i.e. the value $\Delta x \Delta q$) under a constraint of given N_{eff} (5). As is common for such treatments [3, 5–7, 10], one should search for the extremum of the auxiliary functional \mathbb{F} :

$$\mathbb{F}[\rho(\mathbf{X}, \mathbf{X}')] = k^2(\Delta x)^2 + \frac{1}{k^2}(\Delta q)^2 \tag{8}$$

which has the property that its minimum for k (k is a variate scale factor) is attained simultaneously with the minimum of the uncertainty volume

$$\min_k \mathbb{F} = 2\Delta x \Delta q.$$

The substitution of definitions (3) and (4) into (8) shows that the value of \mathbb{F} coincides in form with the mean value of the energy of the s -dimensional symmetrical harmonic oscillator with a Hamiltonian

$$\frac{1}{s} \left(-\frac{1}{k^2} \nabla_{\mathbf{X}}^2 + k^2 \mathbf{X}^2 \right)$$

which is in the mixed state $\rho(\mathbf{X}, \mathbf{X}')$. It is therefore natural in further treatment to represent $\rho(\mathbf{X}, \mathbf{X}')$ as a series expansion in the basis of this Hamiltonian eigenfunction

$$\rho(\mathbf{X}, \mathbf{X}') = \sum_{\mathbf{n}, \mathbf{n}'} a_{\mathbf{n}, \mathbf{n}'} \Psi_{\mathbf{n}}(\mathbf{X}) \Psi_{\mathbf{n}'}(\mathbf{X}') \tag{9}$$

where $\mathbf{n} = (n_1, \dots, n_s)$ is the ‘vector’ index whose components range over non-negative integer numbers

$$\Psi_{\mathbf{n}}(\mathbf{X}) = \prod_{i=1}^s \psi_{n_i}(x_i)$$

$$\psi_n(x) = \sqrt{\frac{k}{2^n n! \sqrt{\pi}}} \exp(-k^2 x^2 / 2) H_n(kx)$$

$H_n(x) = \frac{1}{\sqrt{\pi}} e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$ are Hermite polynomials, the functions $\Psi_{\mathbf{n}}(\mathbf{X})$ are real and orthonormal. In view of what has just been given, the substitution of (9) into (8) gives

$$\mathbb{F} = \sum_{\mathbf{n}} a_{\mathbf{n}, \mathbf{n}} \left(\frac{2}{s} \|\mathbf{n}\| + 1 \right) \quad \|\mathbf{n}\| = \sum_{i=1}^s n_i. \tag{10}$$

Now the additional condition, under which the effective number of states (5) is constant, can be written in the form

$$\mu = \frac{1}{N_{\text{eff}}} = \sum_{\mathbf{n}, \mathbf{n}'} |a_{\mathbf{n}, \mathbf{n}'}|^2 = \text{const} \tag{11}$$

(in statistical optics the parameter μ is frequently referred to as global degree of coherence [5–7], and in literature on quantum state measurement [14] this value is often called ‘purity’). One more constraint on the expansion coefficients $a_{\mathbf{n}, \mathbf{n}'}$ follows from the requirement for the density matrix (6) normalization

$$\sum_{\mathbf{n}} a_{\mathbf{n}, \mathbf{n}} = 1. \tag{12}$$

Without going into details of elementary, although cumbersome intermediate calculations, we now turn to the analysis of the solution obtained. The first and most important consequence

of the minimization of \mathbb{F} (10) in coefficients $a_{n,n}$ is that in the state of least uncertainty $\rho_{\min}(\mathbf{X}, \mathbf{X}')$ all off-diagonal elements of the matrix $[a_{n,n}]$ are equal to zero

$$a_{n,n'} = a_{n,n} \delta_{n,n'}.$$

It means that, in this case, the density operator's eigenfunctions [11] (in optics, the decomposition modes) coincide in form with the eigenfunctions of the energy operator of the multi-dimensional harmonic oscillator. Accordingly, the diagonal elements of matrix $[a_{n,n}]$ are the eigenvalues of the density operator

$$a_{n,n} \Psi_n(\mathbf{X}) = \int d^s X' \rho_{\min}(\mathbf{X}, \mathbf{X}') \Psi_n(\mathbf{X}').$$

These define the probabilities of finding the system in the pure state $\Psi_n(\mathbf{X})$ and satisfy the conditions $\text{Im } a_{n,n} = 0$ (as a consequence of $[a_{n,n}]$ is Hermitian) and $0 \leq a_{n,n} \leq 1$.

It should be noted here that the use of characteristic N_{eff} as a measure of the number of possible pure states of the system is directly related to the operation of the transition to the basis of the density operator's eigenfunctions in representation (9). In the theory of stochastic processes [15] this procedure is referred to as Karhunen–Loève expansion of the ‘correlation function’ $\rho(\mathbf{X}, \mathbf{X}')$ and gives, as is known, the most compact and the most rapidly converging form of this series. Therefore, in this paper (as well as in [10]) under the term ‘state’ what is meant is, as a rule, the eigenstate of the density operator.

The second conclusion, directly following from the form of (10)–(12), is that the weights of states $a_{n,n}$ can only depend on the vector index norm $\|\mathbf{n}\|$, and this dependence is linear and decreasing when $\|\mathbf{n}\|$ increases. Then it is obvious that at any finite value of N_{eff} the number of pure states \mathcal{N} present in the expansion ρ_{\min} with a probability other than zero is also finite. Also in the index domain, the coefficients $a_{n,n} \neq 0$ fill, layer by layer, the interior of an s -dimensional equilateral pyramid with the total number of layers equal to \mathcal{L} ($0 \leq \|\mathbf{n}\| < \mathcal{L}$). The most probable pure state ($\mathbf{n} = 0$) corresponds to the pyramid's vertex. The terms of expansion (9) pertaining to some particular layer are absolutely equivalent and make an equal contribution to the functional \mathbb{F} . Hence the spectrum of the eigenvalues in this mixed state is degenerate, and degeneration multiplicity in each layer is determined by the value of $\|\mathbf{n}\|$ and space dimensionality s

$$g_s(\|\mathbf{n}\|) = \frac{(\|\mathbf{n}\| + s - 1)!}{(\|\mathbf{n}\|)!(s - 1)!}.$$

The total number of significant terms in (9) is given by the relation

$$\mathcal{N}(\mathcal{L}) = \sum_{\|\mathbf{n}\|=0}^{\mathcal{L}-1} g_s(\|\mathbf{n}\|) = \frac{(\mathcal{L} + s - 1)!}{(\mathcal{L} - 1)!s!} = \frac{\mathcal{L}}{s} g_s(\mathcal{L}).$$

Finally, the sought state ρ_{\min} , which realizes the minimum of the generalized uncertainty relation, is described by series (9) with coefficients

$$a_{n,n} = \begin{cases} \frac{1}{\mathcal{N}(\mathcal{L})} \left[1 + ((\mathcal{L} - 1)s - \|\mathbf{n}\|(s + 1)) \frac{\sqrt{(\mathcal{N}(\mathcal{L}) - N_{\text{eff}})(s + 2)}}{\sqrt{N_{\text{eff}}s(\mathcal{L} + s)(\mathcal{L} - 1)}} \right] & 0 \leq \|\mathbf{n}\| < \mathcal{L} \\ 0 & \|\mathbf{n}\| \geq \mathcal{L} \end{cases} \quad (13)$$

and the inequality analogous to (2) takes the form

$$\Delta x \Delta q \geq \frac{2(\mathcal{L} + s - 1)}{2(s + 1)} - \frac{\sqrt{(\mathcal{N}(\mathcal{L}) - N_{\text{eff}})(\mathcal{L} + s)(\mathcal{L} - 1)}}{(s + 1)\sqrt{N_{\text{eff}}s(s + 2)}} = B(N_{\text{eff}}, \mathcal{L}(N_{\text{eff}})). \quad (14)$$

It should also be kept in mind that the quantity \mathcal{L} , a certain positive integer, still remains a free parameter of the task and should be chosen proceeding from the condition of minimality

of the right-hand side of inequality (14). By this means \mathcal{L} will be a certain function of N_{eff} . The above requirements for the expansion coefficients $a_{n,n}$ (13) to be real and positive impose restrictions on allowable values of this parameter, namely the quantity \mathcal{L} should satisfy the following inequalities:

$$N_{\text{eff}} \leq \frac{(\mathcal{L} + s - 1)!}{(\mathcal{L} - 1)!s!} \quad (15)$$

$$N_{\text{eff}} > \frac{(\mathcal{L} + s - 1)!}{(\mathcal{L} - 2)!(s + 1)!} \frac{s + 2}{s + 2(\mathcal{L} - 1)}. \quad (16)$$

There exist certain ranges of values of N_{eff} and s (in particular $s \gg 1$), in which inequalities (15) and (16) determine \mathcal{L} uniquely. But where this is not fulfilled and several integers fall within the interval given by formulae (15) and (16), the sought value of \mathcal{L} turns out to be the largest of them (i.e. the integer nearest to the upper boundary of (16)).

Inequality (14) is more difficult to obtain than the previous inequality (2). On the plane of parameters with coordinates $\Delta x \Delta q$ and N_{eff} it rigorously defines the region of physically realizable states. It is therefore apparent and can easily be proved that one should arrive at the same distributions of the expansion coefficients (13) and the same region (14) by solving the inverse problem of seeking a mixed state with the largest possible value of N_{eff} at a given measure Δv_p of the phase volume occupied by this state.

3. Approximate form of the uncertainty relation

By virtue of the fact that with increasing N_{eff} the quantity \mathcal{L} goes through a set of discrete values, the boundary of the physical region $B(N_{\text{eff}})$ in (14) is not a perfectly smooth curve and is not described by an analytical expression. Even though the numerical calculation of the right-hand side of inequality (14) presents no problem, it would be desirable to have its approximate analytical form in order to analyse the obtained relation and compare it with (2).

To this end we have investigated the behaviour of the function $B(N_{\text{eff}}, \tilde{\mathcal{L}})$ at a given value of N_{eff} , formally, regarding $\tilde{\mathcal{L}}$ as an independent continuous variable. Analytical and numerical calculations show that in the vicinity of the true value of \mathcal{L} the dependence of B on $\tilde{\mathcal{L}}$ is extremely weak. As well, in this region, slightly different neighbouring local minima and maxima for $B(\tilde{\mathcal{L}})$ occur. This gives grounds to use the continuous parameter $\tilde{\mathcal{L}}$ instead of the discrete number \mathcal{L} when approximately describing the boundary of inequality (14). There are several ways of choosing the specific condition determining the value of $\tilde{\mathcal{L}}$.

The following one seems to be fairly simple and logically substantiated. In the expression for expansion coefficients (13), one can formally require a ‘continuous’ transition of its first part to the second one[†], which, after a little algebraic manipulation, yields a transcendental equation for determining $\tilde{\mathcal{L}}$ in the form

$$a_{\tilde{\mathcal{L}}, \tilde{\mathcal{L}}} = 0 \Rightarrow N_{\text{eff}} = \frac{(s + 2)\Gamma(\tilde{\mathcal{L}} + s + 1)}{(s + 2\tilde{\mathcal{L}})(s + 1)!\Gamma(\tilde{\mathcal{L}})} \quad (17)$$

(because $\tilde{\mathcal{L}}$ is continuous the factorials entering into the formula for $\mathcal{N}(\mathcal{L})$ have been replaced here by Euler gamma functions). Accordingly, with such a way of defining \mathcal{L} , the approximate expression for inequality (14) can be written in the parametric form

$$\Delta x \Delta q \geq \frac{s + 2\tilde{\mathcal{L}}(N_{\text{eff}})}{2(s + 2)}. \quad (18)$$

[†] Naturally, in this case, the variable $\|n\|$ in (13) should also be regarded as continuous.

It should be noted that the approximate boundary of the physical domain, given by relations (17) and (18), exactly coincides with the result of the procedure proposed in [3], proceeding from the definition of \mathcal{L} as the upper bound of inequality (16).

Another way of finding a smooth form of the boundary, $B(N_{\text{eff}})$, can be chosen by substituting \mathcal{L} with the value of $\tilde{\mathcal{L}}$ corresponding to $\min_{\tilde{\mathcal{L}}} B(N_{\text{eff}}, \tilde{\mathcal{L}})$. The obtained curve has the advantage that it is wholly located in the physical domain and is tangent to the strict boundary at individual points. Unfortunately, at large s , this method does not lead to analytical formulae.

Finally, according to the above-mentioned property of equivalence of the direct and the inverse problems, the domain of physically realizable states can be represented as an inequality of the largest possible value of N_{eff} at given $\Delta x \Delta q$. On the basis of (17) and (18) it gives the analytical expression

$$N_{\text{eff}} \leq \frac{1}{2\Delta x \Delta q} \frac{\Gamma((s+2)\Delta x \Delta q + s/2 + 1)}{(s+1)! \Gamma((s+2)\Delta x \Delta q - s/2)}. \quad (19)$$

The introduction of the concept of ‘packing coefficient’ $C(s, N_{\text{eff}})$ (whose explicit form is not described by a simple analytical formula) permits us to rearrange relations (14) and (18) into a form similar to (2)[†]. The point of such a transformation is that the main dependence of the generalized uncertainty relation on the parameters of the problem is emphasized, whereas $0 \leq C(s, N_{\text{eff}}) \leq 1$ plays the role of a correction factor with comparatively weak dependence on N_{eff} .

The method of obtaining the approximate expression for the physical domain boundary assumes that the approximation (18) should tend to the exact formula (14) with increasing \mathcal{L} (and, thus N_{eff} as well). In reality, exactly in the limit of $\mathcal{L} \gg 1$, the substitution of a discrete value of \mathcal{L} by a continuous value of $\tilde{\mathcal{L}}$ slightly affects the weights of the individual states in (13). However, even at small $N_{\text{eff}} \gtrsim 1$, the approximate relations (18) and (19) turn out to be in very good accord with the rigorous inequality (14). The reason for this is the above-mentioned local behaviour of $B(\tilde{\mathcal{L}})$.

This is illustrated by plots of exact and approximate dependencies of $C(s, N_{\text{eff}})$ given in figure 1 for some values of s . From this figure it is seen that rather insignificant discrepancies between (14) and (18) take place only at small N_{eff} (near $N_{\text{eff}} \approx 1$), and they are more definite, the greater the problem of the dimension s . At large $N_{\text{eff}} \gg 1$ the approximate solution is equal to the rigorous one. But, with increasing s , quite a good coincidence occurs between approximate relation (18) and the rigorous one (14) at higher values of N_{eff} .

Comparing inequalities (2) and (18), it can be easily shown that (2) is an asymptotic form of (18) (and, consequently, (14)) at $N_{\text{eff}} \rightarrow \infty$, but at $s > 1$ a good approximation of (2) to (14) is only attained at large N_{eff} . At the same time, the refined form of the uncertainty relation (14) correctly describes the whole range of values of N_{eff} and, in particular, the ultimate case of the pure states

$$C(s, N_{\text{eff}} = 1) = 1 \quad (\Delta x \Delta q)^s |_{N_{\text{eff}}=1} \geq \left(\frac{1}{2}\right)^s.$$

The coincidence of the asymptotic form of the packing coefficient $C(s, N_{\text{eff}} \gg 1)$ with formula (7) justifies the use in [10], when proving (2), of the assumption that the Wigner function is non-negative for the density matrix in the state with minimal uncertainty.

[†] Here, as well as in [10], we define the value of $C(s, N_{\text{eff}})$ just as a correction factor of the uncertainty limit for pure states:

$$C(s, N_{\text{eff}}) \equiv \min \frac{(\Delta x \Delta q)^s}{N_{\text{eff}}} \left(\frac{1}{2}\right)^{-s}.$$

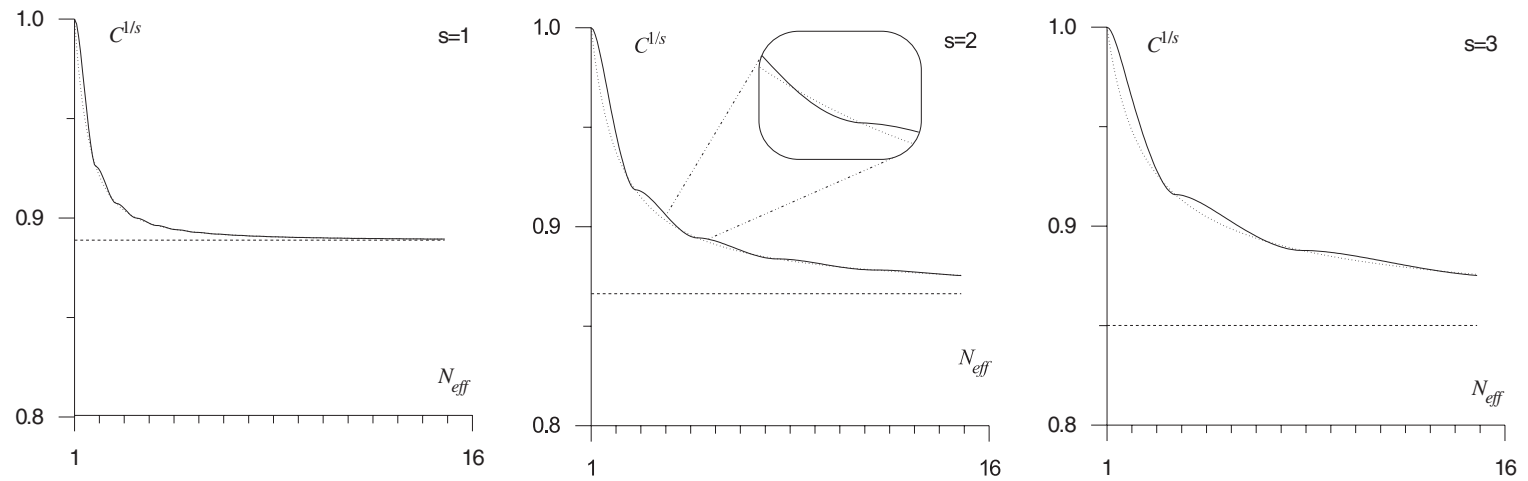


Figure 1. The packing coefficient $C(s, N_{eff})$ versus the effective number of pure states in the statistical mixture (9) for dimensions $s = 1, 2, 3$. Solid curve—strict inequality (14), dotted curve—approximation (18), dashed curve—the asymptotic value $C(s)$ for $N_{eff} \gg 1$ (7).

4. Discussion

At a qualitative level, the treatment of the standard uncertainty relation as a requirement for the phase-space quantization in wave mechanics is rather frequently seen in the literature on physics (see, e.g., [16]). Therefore, it can serve, in some way, as an argument in substantiating the postulate of statistical mechanics on the number of cells in phase space of the system

$$\frac{\mathcal{V}_p(E)}{N} = (2\pi)^s \quad (20)$$

where $\mathcal{V}_p(E)$ is the phase volume of the system with given energy E and N is the number of levels with energies not exceeding E (detailed definitions can be found in [1]). The results of [3, 5–7, 10] and of the present paper open up the possibility of a quantitative comparison of relations (2), (14) and (20).

First of all their similarity lies in the fact that the quantities of the same physical nature enter into the right- and the left-hand sides of expressions (2) and (20) by an identical manner. But the differences between (2) and (20) are far more essential, and there is nothing strange about the differences, if we take into account the way in which the concepts of the phase volume and the number of states are introduced in both cases.

The postulate given by (20) specifies the relation between the energetic structure of the quantum system and the phase space which can be associated with this system in the quasi-classical approximation [1]. Such a relation is assumed to be universal and independent of a particular Hamiltonian of the system. Formula (20) describes the system as a whole and has no connection to its particular physical state.

In contrast, inequalities (2) and (14) just refer to the physical state of the quantum system. Like equation (20), they are explicitly associated with neither the kind of Hamiltonian nor the structure of the energy levels. The only limitation is that, since the basic functions of expansion (9) form a complete set, the wavepackets, satisfying the condition of minimum uncertainty for mixed states, can be constructed only for Hamiltonians whose eigenfunctions also form a complete orthonormal basis. This holds, in particular, for the simplest system of non-interacting particles in a free space. In general, however, the eigenstates of the density operator are not the states with constant energy and, thus, the state with the minimum uncertainty will not be stationary. The only exception is the case of a multi-dimensional harmonic oscillator[†].

It is clear from above that the quantities Δv_p and \mathcal{V}_p , respectively, in equations (2), (14) and (20) not only do not coincide numerically with one another (even for an oscillator), but also have a different operational meaning. A similar statement for a general case is true for the quantities N_{eff} and N as well. So there is nothing surprising in the fact that relation (2) has the form of an inequality, while (20) is an equality. It is possible that a better correlation between (2) and (20) can be attained by using other, alternative formulations of the uncertainty principle [3].

It is worthwhile drawing attention to one more difference between (2) and (20). In the postulate given by (20), the phase-space volume of any quantum cell is constant and varies with a change of dimensionality in a strictly definite way—as $(\text{const})^s$. In contrast, in the generalized uncertainty relation (14), the minimal specific phase volume (i.e. the volume per one effective pure state of the system) depends on the value of N_{eff} . As seen from figure 1, the quantity $C(s, N_{\text{eff}})$ is less than unity and monotonically decreases with increasing N_{eff} . It means that as the number of pure states involved in the statistical mixture for $\rho(\mathbf{X}, \mathbf{X}')$ rises, there is a gain in the packing density of the states.

[†] One should choose the unit of length or, what amounts to the same thing, the value of parameter k to make \mathbb{F} the expectation value of the harmonic oscillator Hamiltonian.

In paper [10] it was also noted that the effect of increasing packing with the augment of the configurational space dimension s , is expressed by the inequality

$$C(ks) \leq C(s)^k \quad (k \geq 1).$$

From (19) one can easily see that an analogous relation for the coefficients $C(s, N_{\text{eff}})$ in the general case does not hold and this property of packing is asymptotic, i.e. in each particular case (s, k) it takes place beginning with sufficiently large N_{eff} . The reason for this should be sought in the peculiarities of behaviour of degeneration multiplicity $g_s(\|\mathbf{n}\|)$ as a function of $\|\mathbf{n}\|$ and s . It is natural to suppose that the decrease in the coefficient $C(s)$ with increasing s is due to the rapid growth of the value of g_s , and for different dimensions it would be proper to compare the packing at equal values of the parameter \mathcal{L} rather than N_{eff} .

In this context, it is interesting to consider the case of the density matrix (9) with equal weights of all pure states $a_{n,n'} = \delta_{n,n'}/\mathcal{N}$ involved, that somewhat resembles the definition of N in (20). For this situation all methods of determining N_{eff} give $N_{\text{eff}} = \mathcal{N}$ and, although the minimum of the uncertainty relation (14) for such a system is slightly exceeded, both the above properties of the packing coefficient are fulfilled.

It is necessary to discuss also the question of what physical meaning is attributed to the arguments (coordinates) entering into the density matrix $\rho(\mathbf{X}, \mathbf{X}')$. These may be either the coordinates of one particle in the real three-dimensional space or the coordinates of several particles in the configurational space, or both at once. The above solution for the density matrix in the state with minimal uncertainty is invariant under any permutation of its arguments ($x_i \Leftrightarrow x_j$). For the real coordinates of one particle it means that the corresponding wavepacket possesses some rotational symmetry in three-dimensional space. But when arguments being interchanged correspond to several particles, it is necessary to take into account the properties of their permutational symmetry. For indistinguishable identical particles the state with such a symmetry can only be realized in the case when these particles follow Bose statistics. In contrast, identical particles with Fermi statistics cannot be described by a density matrix of the form (9) with coefficients given by (13). The state with minimal uncertainty for fermions should be sought, from the very beginning, in the class of wavefunctions antisymmetrized in permutations of arguments, which, naturally, should lead to a result drastically different from (14).

Finally, we shall briefly mention that our results are applicable to density matrices of any origin. For example, all the considerations will remain valid for a marginal density matrix of a subsystem when the whole system is in an entangled state. The question of whether all subsystems can achieve a minimum of the uncertainty principle is not trivial, and its detailed discussion will be the subject of another paper.

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References

- [1] Landau L D and Lifshitz E M 1977 *Quantum Mechanics: Non-Relativistic Theory* (Oxford: Pergamon)
- [2] Kholevo A S 1982 *Probabilistic and Statistical Aspects of Quantum Theory* (Amsterdam: North-Holland)
- [3] Dodonov V V and Man'ko V I 1987 *Trudy FIAN* **183** 5

- [4] Stoler D and Newman S 1972 *Phys. Lett. A* **38** 433
- [5] Bastiaans M J 1982 *J. Opt. Soc. Am.* **72** 1441
- [6] Bastiaans M J 1983 *J. Opt. Soc. Am.* **73** 251
- [7] Bastiaans M J 1984 *J. Opt. Soc. Am. A* **1** 711
- [8] Bochvar D A, Stankevich I V and Chistiakov A L 1974 *Usp. Khimii (Adv. Chem.)* **43** 655
- [9] Trifonov D A and Donev S G 1998 *J. Phys. A: Math. Gen.* **31** 8041
- [10] Karelin N V and Lazaruk A M 1998 *Theor. Math. Phys.* **117** 1447
- [11] Feynman R P 1972 *Statistical Mechanics* (Reading, MA: Addison-Wesley)
- [12] Landau L D and Lifshitz E M 1980 *Statistical Physics Part I* (Oxford: Pergamon)
- [13] Karelin N V and Lazaruk A M 1997 *Radiophys. Quantum Electron.* **40** 603
- [14] Leonardt U 1997 *Measuring of the Quantum State of Light* (Cambridge: Cambridge University Press)
- [15] Fu K 1968 *Sequential Methods in Pattern Recognition and Machine Learning* (New York: Academic)
- [16] Kubo R 1965 *Statistical Mechanics* (Amsterdam: North-Holland)